Ground-state degeneracies of Ising spin glasses on diamond hierarchical lattices

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The total number of ground states for short-range Ising spin glasses, defined on diamond hierarchical lattices of fractal dimensions $d=2$, 3, 4, 5, and 2.58, is estimated by means of analytic calculations (three last hierarchy levels of the $d=2$ lattice) and numerical simulations (lower hierarchies for $d=2$ and all remaining cases). It is shown that in the case of continuous probability distributions for the couplings, the number of ground states is finite in the thermodynamic limit. However, for a bimodal probability distribution ($\pm J$ with probabilities p and $1-p$, respectively), the average number of ground states is maximum for a wide range of values of p around $p=\frac{1}{2}$ and depends on the total number of sites at hierarchy level n, $N^{(n)}$. In this case, for all lattices investigated, it is shown that the ground-state degeneracy behaves like $\exp[h(d)N^{(n)}]$, in the limit $N^{(n)}$ large, where $h(d)$ is a positive number which depends on the lattice fractal dimension. The probability of finding frustrated cells at a given hierarchy level n, $F^{(n)}(p)$, is calculated analytically (three last hierarchy levels for $d=2$ and the last hierarchy of the $d=3$ lattice, with $0\leqslant p\leqslant 1$), as well as numerically (all other cases, with $p=\frac{1}{2}$). Except for $d=2$, in which case $F^{(n)}(\frac{1}{2})$ increases by decreasing the hierarchy level, all other dimensions investigated present an exponential decrease in $F^{(n)}(\frac{1}{2})$ for decreasing values of n. For $d=2$ our results refer to the paramagnetic phase, whereas for all other dimensions considered [which are greater than the lower critical dimension d_l ($d_l\approx 2.5$)], our results refer to the spin-glass phase at zero temperature; in the latter cases $h(d)$ increases with the fractal dimension. For $n\gg 1$, only the last hierarchies contribute significantly to the ground-state degeneracy; such a dominant behavior becomes stronger for high fractal dimensions. The exponential increase of the number of ground states with the total number of sites is in agreement with the mean-field picture of spin glasses. [S1063-651X(99)07910-6]

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I. INTRODUCTION

A satisfactory understanding of short-range spin glasses [1–3] is still missing, in spite of a considerable amount of effort dedicated to this problem. Most of the definite results are at mean-field level, based on the solution of the infinite-range-interaction Sherrington-Kirkpatrick model [4]. Such a model predicts a spin-glass phase characterized by a complicated free-energy landscape with many minima, properly described by an infinite number of order parameters, i.e., an order-parameter function [5]. An equivalent approach, as proposed by Thouless, Anderson, and Palmer (TAP) [6], should be to solve the complete set of mean-field equations, the so-called TAP [6] equations. This turns out to be a difficult task, since the average number of solutions of the TAP equations grows exponentially with the total number of sites N [7,8],

$$[N_S]_J \sim \exp(\alpha N), \tag{1.1}$$

where $[\ ]_J$ denotes an average over the disorder and $\alpha=\alpha(h,T)$ is a function of both external field h and temperature T. In the paramagnetic phase, $\alpha(h,T)=0$, whereas $\alpha(0,0)\cong 0.20$ [7,8]. These solutions may not be all physical (some of them may be unstable) throughout the spin-glass phase, though one can show that they are all stable at $T=0$ [7]. If one associates each $T=0$ solution to a ground state, then mean field predicts an average number of ground states which diverges exponentially, in the thermodynamic limit, according to Eq. (1.1).

Whether such a mean-field picture prevails in real (short-range) systems is a very controversial point [1]. In particular, the ground-state degeneracy and a proper understanding of the phase-space structure are questions which remain unanswered. Unfortunately, exact results for short-range spin glasses are very scarce; such systems on Bravais lattices are, most of the time, investigated by numerical methods, like power-series expansions and simulations.

The search for the ground states of short-range spin glasses has attracted the attention of many workers through the last two decades [9–28], leading to the development of a large variety of algorithms. It is well accepted nowadays that due to the existence of many low-energy metastable states, separated by energy barriers, the usual Monte Carlo method finds serious difficulties in providing the true ground states of spin glasses; one needs alternative numerical (or analytical) techniques to work at zero temperature. The most intuitive approach, known as the minimal matching of frustrated plaquettes, is based on the idea of Toulouse [9] and consists in defining a ground-state configuration through a pairwise connection of the frustrated plaquettes by strings, in such a way as to minimize the total sum of string lengths. A direct counting of ground states through this method is feasible for two-dimensional lattices [10–12,16], but becomes computa-

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tionally impracticable for three-dimensional systems [10]. Other methods, like mappings into combinatorial optimization problems [13,14,20,24], graph expansions [18], biologically motivated algorithms [21–23,26–28], and Monte Carlo multicanonical techniques [17], have produced significant progress in the knowledge of spin-glass ground states. The multicanonical technique of Berg, Hansmann, and Celik [17] is probably the first Monte Carlo method to generate successfully the ground states of the three-dimensional $\pm$J spin glass; their data are consistent with a simple ground-state structure, typical of the droplet scaling ansatz [29–32], although a mean-field-like picture is not ruled out. The recent algorithm introduced by Hartmann [22], combining genetic algorithms with cluster-exact approximations, yields ground states for three-dimensional spin glasses within a polynomial time [$\sim O(N^3)$]. Applying this method for $\pm$J spin glasses, Hartmann found evidence of a nontrivial ground-state structure (typical of mean-field theory) on a cubic lattice [26], but not on a square lattice [28]. Therefore, the study of the spin-glass ground states appears as a very promising tool for a correct understanding of the structure of the spin-glass phase.

Although hierarchical lattices are not Bravais lattices, they are much easier to handle (under real-space renormalization group), in such a way that exact results [33,34] may be obtained for short-range systems. For pure systems defined on Bravais lattices, certain renormalization-group equations are obtained through decimation of spins; in the corresponding hierarchical lattices, such a procedure is exact for models with discrete classical spin variables, leading, within a few renormalization steps, to nonproliferated renormalization-group recursion relations connecting two successive hierarchy levels. However, for random systems, the recursion relations depend on random variables, described by probability distributions. The process of following the evolution of the probability distributions as the hierarchy levels change may become a difficult task to carry out exactly; most of the time one makes use of numerical methods. Therefore, real-space renormalization-group techniques, for random systems on hierarchical lattices, are usually approximate; however, depending on the kind of approximations involved, the corresponding renormalization-group recipe may come up as a good approximation for the hierarchical lattice.

For integer fractal dimensions, hierarchical lattices may be considered as approximations of Bravais lattices and should lead to some physical insight into the behavior of real spin glasses. In spite of its simplicity, the diamond hierarchical lattices (DHLs) have been successful in estimating lower critical dimensions and critical temperatures of Ising spin glasses [35], almost a decade before their confirmation through power-series expansions [36] and numerical simulations [37] on Bravais lattices.

In this paper we estimate the average number of ground states for nearest-neighbor-interaction Ising spin glasses, defined on DHLs of fractal dimensions $d=2$, 3, 4, 5, and 2.58. For that, we make use of analytic calculations, as well as of numerical simulations. In the next section we define the model and the formalism employed in the calculation of the ground-state degeneracy. In Sec. III we present the results for a DHL with fractal dimension $d=2$, and, in Sec. IV, those for dimensions $d=3$, 4, 5, and 2.58. Finally, in Sec. V we present our conclusions.

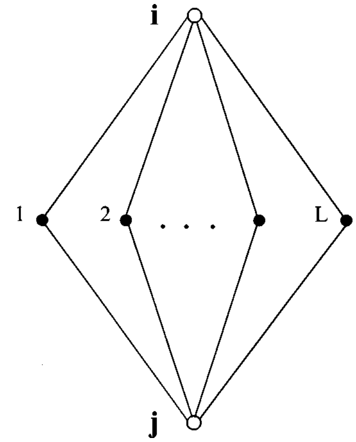

FIG. 1. The basic cell of the diamond hierarchical lattice with fractal dimension $d=(\ln 2L)/(\ln 2)$. The solid circles denote the internal sites, whereas the open ones represent the external sites (connected to other cells of the lattice).

II. MODEL AND FORMALISM

Let us consider the Ising spin glass, defined through the Edwards-Anderson Hamiltonian [38],

$$\mathcal{H}=-\sum_{\langle ij\rangle} J_{ij}S_iS_j \quad (S_i=\pm 1), \tag{2.1}$$

where the J_{ij}'s represent random quenched coupling constants following a certain probability distribution $P(J_{ij})$. The sum $\Sigma_{\langle ij\rangle}$ applies to nearest-neighbor pairs of spins on a DHL, generated in such a way that at each step a bond is replaced by a diamondlike cell containing L parallel branches, each with two bonds in series (scaling factor $b=2$), as shown in Fig. 1; its fractal dimension is $d=(\ln 2L)/(\ln 2)$. At the zeroth hierarchy level one starts with a single bond, which is replaced by a single cell at hierarchy level $n=1$. By following this procedure, one gets for the DHL at its nth hierarchy level the total number of cells ($N_c^{(n)}$) and sites ($N^{(n)}$) given, respectively, by

$$N_c^{(n)}=(2L)^{n-1}\equiv(2^d)^{n-1}, \tag{2.2a}$$

$$N^{(n)}=2+L\,\frac{(2L)^n-1}{2L-1}, \tag{2.2b}$$

and in the limit $n\gg 1$, one has

$$N^{(n)}\approx\frac{L}{2L-1}(2L)^n=\frac{L}{2L-1}2LN_c^{(n)}=\frac{L}{2L-1}(2L)^2N_c^{(n-1)}$$
$$=\cdots. \tag{2.2c}$$

For integer values of d, the results obtained on DHLs may be considered as approximations for the corresponding Bravais lattices.

We start with a DHL at the nth hierarchy level with $P^{(n)}(J_{ij}^{(n)})$ as either a continuous or a bimodal one,

$$P^{(n)}(J_{ij}^{(n)})=p\,\delta(J_{ij}^{(n)}-J)+(1-p)\,\delta(J_{ij}^{(n)}+J). \quad (2.3)$$

We will restrict our analysis to zero temperature. The multiplicity of ground states is produced by frustration [9] effects; let us now adapt this concept for the cell in Fig. 1. We say that a given cell is frustrated whenever a bond configuration leads to at least one branch α ($\alpha=1,2,...,L$) with an *arbitrariness* in the state of its internal spin, i.e., the total energy of the cell remaining unchanged under the spin flip. Therefore, each branch in this situation will contribute with a factor of 2 for the multiplicity of states of the cell; if a cell contains α branches with such arbitrariness, it will contribute for the total number of ground states of the DHL with a multiplicity factor,

$$g_{\alpha}=2^{\alpha} \quad (\alpha=1,2,...,L). \quad (2.4)$$

From now on, a given frustrated cell, associated with a multiplicity factor g_{α}, will be referred to as a cell of the type α (or simply, as an α cell). The trivial case $g_0=1$ (nonfrustrated cell) is obviously excluded from the set of α cells defined in Eq. (2.4).

For a given bond configuration defined by $P^{(n)}(J_{ij}^{(n)})$, one may count partially the average number of ground states for a DHL at the nth level, by fixing the terminal spins of each unit cell [i.e., the spins that connect the cell to other ones (open circles in Fig. 1)] and counting the distribution of α cells. In this case, only the internal spins (full circles in Fig. 1) of each unit cell will contribute to the multiplicity of ground states. We shall denote the number representing this partial counting by $[\Gamma^{(n)}]_{J^{(n)}}$, where $[\ ]_{J^{(n)}}$ represents an average over the coupling probability distribution at level n, $P^{(n)}(J_{ij}^{(n)})$. For a hierarchical lattice at its nth level, all unit cells present one terminal spin that belongs to the $(n-1)$th level, whereas the other one is associated with a lowest-level hierarchy; the terminal spins that belong to the $(n-1)$th level become internal ones at level $n-1$, whereas those of the lowest-level hierarchies become terminal spins at level $n-1$; a similar procedure holds for the $(n-2)$th level, and so on. Due to this, in order to calculate the average number of ground states at level n, we will take advantage of the real-space renormalization-group approach for DHLs. In the case of spin glasses, throughout the renormalization process the probability distribution will change its shape, evolving to particular distributions according to the respective phase, or it will approach a ''fixed-point'' distribution, characteristic of the corresponding critical frontier [35]. The decimation of the internal spins of the cell in Fig. 1 leads to a renormalized coupling between sites i and j [35],

$$J'_{ij}=\frac{1}{2}\sum_{l=1}^{L}\,(|J_{il}+J_{lj}|-|J_{il}-J_{lj}|), \quad (2.5)$$

where J_{il} and J_{lj} represent the two original coupling constants associated with a given branch l connecting the external sites of the cell. Following the arguments above, the average number of ground states at hierarchy level n may be written as

$$[N_{\rm GS}^{(n)}]_{J^{(n)}}=[\Gamma^{(n)}]_{J^{(n)}}[\Gamma^{(n-1)}]_{J^{(n-1)}} \times[\Gamma^{(n-2)}]_{J^{(n-2)}}\cdots[\Gamma^{(1)}]_{J^{(1)}}A, \quad (2.6)$$

where the factor A corresponds to the number of states associated to a single bond (hierarchy level 0).

Let us now describe how to calculate $[\Gamma^{(k)}]_{J^{(k)}}$ ($k=n,n-1,n-2,...,1$). First, let $F_{\alpha}^{(k)}$ be the probability of finding a frustrated cell of the type α, at the hierarchy level k; obviously, $F_{\alpha}^{(k)}$ depends on the parameters of the distribution $P^{(k)}(J_{ij}^{(k)})$ [which, in turn, may depend on the parameters of the initial distribution $P^{(n)}(J_{ij}^{(n)})$], e.g., in the $\pm J$ case [Eq. (2.3)], $F_{\alpha}^{(k)}\equiv F_{\alpha}^{(k)}(p)$. The average number of α cells in the hierarchy level k is given by

$$\phi_{\alpha}^{(k)}=N_c^{(k)}F_{\alpha}^{(k)}, \quad (2.7)$$

where $N_c^{(k)}$ is the total number of unit cells, as defined in Eq. (2.2a). Obviously, the total probability of finding a frustrated cell and the average number of frustrated cells at hierarchy level k are given, respectively, by

$$F^{(k)}=\sum_{\alpha=1}^{L}\,F_{\alpha}^{(k)}, \quad \phi^{(k)}=\sum_{\alpha=1}^{L}\,\phi_{\alpha}^{(k)}=N_c^{(k)}F^{(k)}. \quad (2.8)$$

If one includes the nonfrustrated cells (associated probability $F_0^{(k)}$) one gets the normalization condition $\Sigma_{\alpha=0}^{L}F_{\alpha}^{(k)}=1$. Hence, the average number of ground states of the DHL at hierarchy level k, obtained by fixing the terminal spins of each unit cell, is given by

$$[\Gamma^{(k)}]_{J^{(k)}}=\prod_{\alpha=1}^{L}\,(g_{\alpha})^{\phi_{\alpha}^{(k)}}=(\gamma^{(k)})^{\phi^{(k)}}, \quad (2.9)$$

where

$$\gamma^{(k)}\equiv\prod_{\alpha=1}^{L}\,(g_{\alpha})^{(\phi_{\alpha}^{(k)}/\phi^{(k)})}=\prod_{\alpha=1}^{L}\,(g_{\alpha})^{(F_{\alpha}^{(k)}/F^{(k)})}. \quad (2.10)$$

One may define the ground-state complexity [39],

$$I=k_B\ln[N_{\rm GS}^{(n)}]_{J^{(n)}}=k_B\left(\ln A+\sum_{k=1}^{n}\,\ln[\Gamma^{(k)}]_{J^{(k)}}\right) =k_B\left(\ln A+\sum_{k=1}^{n}\,\phi^{(k)}\ln\gamma^{(k)}\right), \quad (2.11)$$

which is an extensive quantity if

$$[N_{\rm GS}^{(n)}]_{J^{(n)}}\sim\exp[h(d)N^{(n)}], \quad (2.12)$$

with $h(d)$ being a constant for each DHL of fractal dimension d. The quantity $k_B\ln\gamma^{(k)}[=(k_B\ln[\Gamma^{(k)}]_{J^{(k)}})/\phi^{(k)}]$ may be identified as the partial contribution, per frustrated cell, to the ground-state complexity, i.e., as the partial complexity per frustrated cell; from now on, we shall refer to $\gamma^{(k)}$ as the average number of ground states per frustrated cell (terminal spins fixed) at hierarchy level k.

The main difficulty of the present approach consists in calculating $[\Gamma^{(k)}]_{J^{(k)}}$, or equivalently, the probabilities $F_{\propto}^{(k)}$, from which one may obtain the average number of α cells using Eq. (2.7). Although this can be done analytically in some cases (e.g., for the last hierarchies of a $d=2$ $\pm$J DHL), it may become a hard task for DHLs of high fractal dimensions. However, such quantities can be easily computed through a numerical simulation, as we describe now. For that purpose, one should generate a large number of different disorder samples $\{J_{ij}^{(k)}\}$ of a given DHL and count, for each sample, the number of α cells in order to compute $F_{\alpha}^{(k)}$ and consequently, all the quantities defined in Eqs. (2.7)–(2.10). In addition to that, one also needs to renormalize the probability distribution, at each hierarchy level, using the recursive relation (2.5); starting the simulation at the nth hierarchy level [with a given distribution $P^{(n)}(J_{ij}^{(n)})$], one generates $P^{(n-1)}(J_{ij}^{(n-1)})$, which should be stored to be used at hierarchy $n-1$, which in its turn, will generate $P^{(n-2)}(J_{ij}^{(n-2)})$, and so on. Instead of working with the whole DHL, we generate at each hierarchy level k a number M of independent unit cells, with coupling constants distributed according to $P^{(k)}(J_{ij}^{(k)})$, from which one may compute all the quantities in Eqs. (2.7)–(2.10), and $P^{(k-1)}(J_{ij}^{(k-1)})$. In order to get a good statistics on each hierarchy level, we repeat this process for N_s different samples (different sequences of random numbers). If one starts with a bimodal probability distribution at the nth hierarchy level, a proliferation of δ functions will occur throughout the renormalization process; this method allows one to compute easily the weights of the δ functions at each hierarchy level k, in order to obtain $P^{(k-1)}(J_{ij}^{(k-1)})$. One expects that if M and N_s are large enough, such a procedure should give results similar to those obtained by considering averages over disorder samples of the whole DHL. For the results that will be presented in the next sections, we have used $M=10^7$ and $N_s=200$. For a symmetric bimodal distribution $P^{(n)}(J_{ij}^{(n)})$ we have performed some analytic calculations in DHLs of fractal dimensions $d=2$ and $d=3$; it was possible to check the accuracy of our simulations in such cases. The scheme described above yielded estimates for $F^{(k)}$, $\gamma^{(k)}$, and the δ weights, in agreement up to three decimal digits with the analytic results.

In what follows we shall treat the case $d=2$ separately from those where d is greater than the lower critical dimension d_l ($d_l \approx 2.5$ [40–42]). In the former, there is no spin-glass phase [31,35–37,40,41], in such a way that any distribution will converge, under the renormalization process, to a single δ function centered at the origin (characteristic of a paramagnetic phase). In the latter ones, all distributions will approach, after many iterations, a continuous distribution with increasing variance, associated with the spin-glass phase [40].

III. CASE $d=2$

Let us first consider an arbitrary continuous distribution $P^{(n)}(J_{ij}^{(n)})$. It is clear that for a frustration to occur one must have a branch l with strictly $|J_{il}|=|J_{lj}|$; since this never happens for continuous probability distributions, there will be no frustrated cells ($F_{\alpha}^{(k)}=0$; $\alpha=1,2,...,L$) at the highest hierarchies (e.g., n, $n-1$, and $n-2$). However, in the lowest-level hierarchies, the distribution will approximate a δ function centered at the origin and one will have multiplicity of states produced by zero coupling constants. Since such frustration effects only occur for $k \ll n$ ($n \gg 1$), and considering the fact that the number of sites of the high hierarchies is much larger than the ones of the low hierarchies ($N^{(n)} \sim 2^{2n}$), one expects to find $\phi^{(k)}$ finite in this limit. For a null single bond (hierarchy 0), one has the factor $A=4$. Putting all these results together into Eq. (2.11), one will get a finite complexity, negligible in the thermodynamic limit.

For the remainder of this section we will be restricted to the bimodal probability distribution of Eq. (2.3). For $k=n$, there are only two types of frustrated cells (each containing one branch with arbitrariness), in such a way that

$$F^{(n)}(p) \equiv F_1^{(n)}(p) = 4p^3(1-p) + 4p(1-p)^3, \quad g_1^{(n)} = 2,$$

which lead to the average number of frustrated cells,

$$\phi^{(n)}(p) \equiv \phi_1^{(n)}(p) = 2^{2(n-1)}\{4p^3(1-p)+4p(1-p)^3\}, \tag{3.1}$$

and to $\gamma^{(n)}=2$ (independent of p). In order to proceed to hierarchy level $n-1$ one needs the renormalized probability distribution,

$$\begin{aligned} p^{(n-1)}(J_{ij}^{(n-1)}) = {} & P_{2J}^{(n-1)}(p)\,\delta(J_{ij}^{(n-1)}-2J) + P_0^{(n-1)} \\ & \times (p)\,\delta(J_{ij}^{(n-1)}) + P_{-2J}^{(n-1)}(p) \\ & \times \delta(J_{ij}^{(n-1)}+2J), \end{aligned} \tag{3.2}$$

which has turned into a three-δ form, with

$$P_{2J}^{(n-1)}(p) = p^4 + 2p^2(1-p)^2 + (1-p)^4, \tag{3.3a}$$

$$P_0^{(n-1)}(p) = 4p^3(1-p) + 4p(1-p)^3, \tag{3.3b}$$

$$P_{-2J}^{(n-1)}(p) = 4p^2(1-p)^2. \tag{3.3c}$$

The analysis of all frustrated cells associated to Eq. (3.2) leads to

$$\begin{aligned} F^{(n-1)}(p) = {} & 4[P_{2J}^{(n-1)}(p)]^3 P_{-2J}^{(n-1)}(p) + 4P_{2J}^{(n-1)}(p)[P_{-2J}^{(n-1)} \\ & \times (p)]^3 + 2[P_{2J}^{(n-1)}(p)]^2[P_0^{(n-1)}(p)]^2 \\ & + 4P_{2J}^{(n-1)}(p)[P_0^{(n-1)}(p)]^2 P_{-2J}^{(n-1)}(p) \\ & + 2[P_0^{(n-1)}(p)]^2[P_{-2J}^{(n-1)}(p)]^2 + 4P_{2J}^{(n-1)}(p) \\ & \times [P_0^{(n-1)}(p)]^3 + 4[P_0^{(n-1)}(p)]^3 P_{-2J}^{(n-1)}(p) \\ & + [P_0^{(n-1)}(p)]^4, \end{aligned}$$

which become, after substituting Eqs. (3.3) and using the software MATHEMATICA 3.0 to simplify,

$$\begin{aligned} F^{(n-1)}(p) = {} & -16p^2(1-3p+4p^2-2p^3)^2(-3+8p-16p^2 \\ & -16p^3+120p^4-224p^5+224p^6-128p^7 \\ & +32p^8). \end{aligned} \tag{3.4}$$

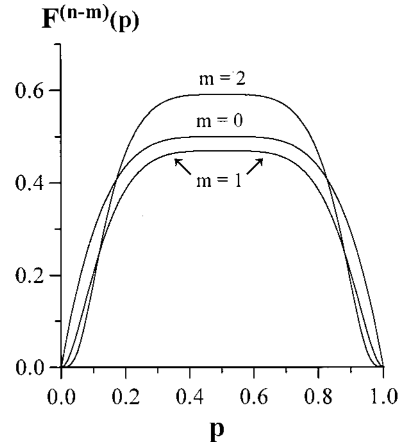

FIG. 2. The total probability of finding frustrated cells, for the DHL of fractal dimension $d=2$, at hierarchy levels $k=n-m$ ($m=0,1,2$), as a function of the initial ferromagnetic coupling weight p of the bimodal distribution $P^{(n)}(J_{ij}^{(n)})$.

One has $g_1^{(n-1)}=2$ for all frustrated cells, except for the case of a cell with all four bonds $J_{ij}^{(n-1)}=0$ $\{F_2^{(n-1)}(p)=[P_0^{(n-1)}(p)]^4;\ g_2^{(n-1)}=4\}$.

For $k=n-2$, the probability distribution is composed by five δ's,

$$P^{(n-2)}(J_{ij}^{(n-2)})=P_{4J}^{(n-2)}(p)\,\delta(J_{ij}^{(n-2)}-4J)+P_{2J}^{(n-2)}(p)\,\delta(J_{ij}^{(n-2)}-2J)+P_0^{(n-2)}(p)\,\delta(J_{ij}^{(n-2)})+P_{-2J}^{(n-2)}(p)\,\delta(J_{ij}^{(n-2)}+2J)+P_{-4J}^{(n-2)}(p)\,\delta(J_{ij}^{(n-2)}+4J), \tag{3.5}$$

whose weights are functions of those at hierarchy level $n-1$ [Eqs. (3.3)]; after using MATHEMATICA 3.0 to simplify,

$$P_{4J}^{(n-2)}(p)=(1-8p+32p^2-80p^3+152p^4-224p^5+224p^6-128p^7+32p^8)^2, \tag{3.6a}$$

$$P_{2J}^{(n-2)}(p)=-16p(-1+13p-88p^2+404p^3-1408p^4+3920p^5-8928p^6+16\,840p^7-26\,400p^8+34\,144p^9-35\,712p^{10}+29\,248p^{11}-17\,920p^{12}+7680p^{13}-2048p^{14}+256p^{15}), \tag{3.6b}$$

$$P_0^{(n-2)}(p)=16p^2(1-3p+4p^2-2p^3)^2(5-24p+96p^2-240p^3+456p^4-672p^5+672p^6-384p^7+96p^8), \tag{3.6c}$$

$$P_{-2J}^{(n-2)}(p)=-128p^3(-1+3p-4p^2+2p^3)^3(1-2p+6p^2-8p^3+4p^4), \tag{3.6d}$$

$$P_{-4J}^{(n-2)}(p)=64(-1+p)^4p^4(1-2p+2p^2)^4. \tag{3.6e}$$

Similarly to what was done for hierarchy levels n and $n-1$, one may calculate $F^{(n-2)}(p)$. In Fig. 2 we exhibit the p dependence of the total probability for finding frustrated cells at hierarchy levels $k=n-m$ ($m=0$, 1, and 2), obtained through the method described above. When $p\to 1$ (or equivalently, $p\to 0$) one trivially obtains $F^{(n-m)}(p)\to 0$ i.e., $\phi^{(n-m)}\to 0$, whereas $A=2$; this leads to the expected result $[N_{GS}^{(n)}]_{J^{(n)}}=2$. For a wide interval of p around $p=\frac{1}{2}$, one notices that the total probability of finding frustrated cells decreases as one goes from hierarchy level n to $n-1$, a direct consequence of the appearance of the δ function at the origin [see Eq. (3.2)]; however, if $m>1$ this probability always increases. One should notice that the curves in Fig. 2 present flat maxima, indicating that the maximum ground-state degeneracy occurs within a wide range of values of p around $p=\frac{1}{2}$. In the discussion which follows we will be restricted to $p=\frac{1}{2}$, although our results apply for a wider p interval.

Even if the number of δ functions grows very fast in the next steps, one may easily see that the one centered at $J_{ij}^{(k)}=0$ will prevail among the others, for a wide range of values of p around $p=\frac{1}{2}$. Indeed, one has the weights $P_0^{(n)}(\frac{1}{2})=0$, $P_0^{(n-1)}(\frac{1}{2})=\frac{1}{2}$, and $P_0^{(n-2)}(\frac{1}{2})=\frac{19}{32}=0.593\,75$.

We have performed numerical simulations to compute $P_0^{(n-m)}(1/2)$, $F_\alpha^{(n-m)}(1/2)$ [and consequently, all the quantities defined in Eqs. (2.7)–(2.10)] for systems up to $m=10$. The numerical estimates for $m=0$, 1, and 2 are in full agreement (up to three decimal digits) with the exact results obtained above. The results of our simulations are exhibited in Figs. 3(a)–3(c); one sees that after a few renormalization steps (roughly, six iterations) the probability distribution approaches $P^{(k)}(J_{ij}^{(k)})=\delta(J_{ij}^{(k)})$, characteristic of a paramagnetic phase. In this limit, only the $\alpha=2$ cells will contribute, i.e., $F^{(k)}(\frac{1}{2})\equiv F_2^{(k)}(\frac{1}{2})=1$ and so,

$$\phi^{(k)}=N_c^{(k)}=2^{2(k-1)}, \quad \gamma^{(k)}=4. \tag{3.7}$$

The complexity in Eq. (2.11) becomes

$$I=k_B\left(\ln A+\sum_{k=n-6}^{n}\phi^{(k)}\ln\gamma^{(k)}+\sum_{k=1}^{n-7}N_c^{(k)}\ln 4\right), \tag{3.8}$$

and using the analytical results ($k=n, n-1, n-2$), as well as those obtained from the numerical simulations ($k<n-2$),

$$\sum_{k=n}^{n-6}\phi^{(k)}\ln\gamma^{(k)}=0.346\,57N_c^{(n)}+0.368\,24N_c^{(n-1)}+0.497\,16N_c^{(n-2)}+(0.701\,52\pm 0.000\,06)N_c^{(n-3)}+0.984\,77N_c^{(n-4)}+1.254\,53N_c^{(n-5)}+1.373\,28N_c^{(n-6)}.$$

In the equation above, we have written the error bars due to the numerical simulations for the case $k=n-3$ only; as will be seen, the error bars for $k<n-2$ will not affect the most significant digits of the final result. In the limit $n\gg 1$ one has, from Eq. (2.2c),

$$N^{(n)}\approx\tfrac{2}{3}(4)^n=\tfrac{8}{3}N_c^{(n)}=\tfrac{32}{3}N_c^{(n-1)}=\cdots,$$

and then

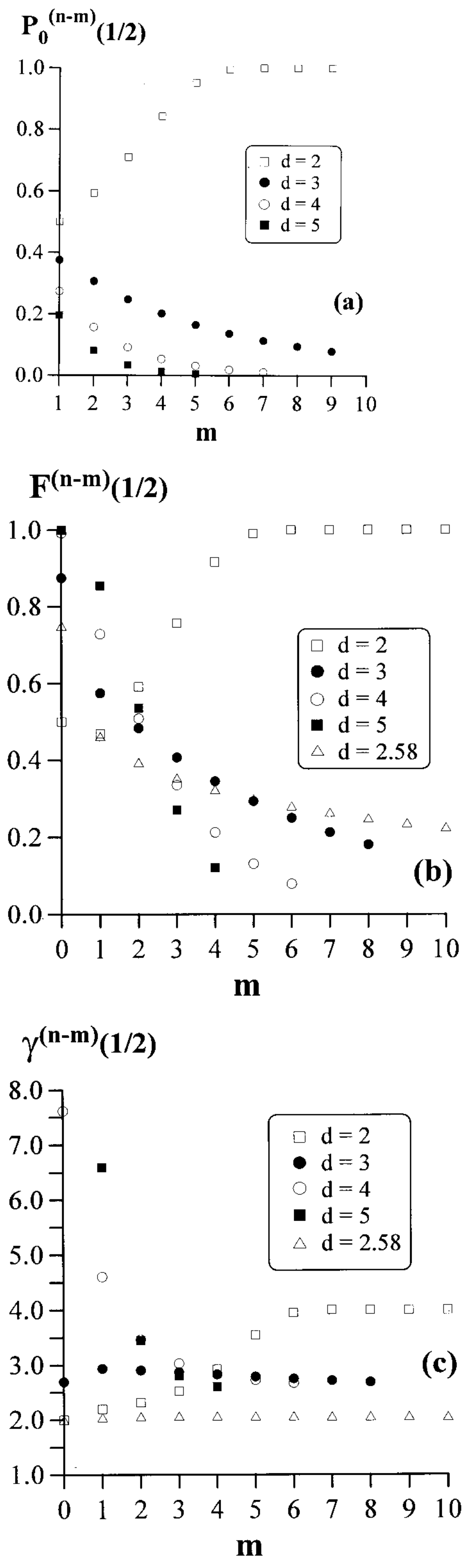

FIG. 3. Results from numerical simulations of DHLs of several fractal dimensions d, at hierarchy levels $k=n-m$. At the nth hierarchy level, a symmetric bimodal ($p=\frac{1}{2}$) probability distribution was used. (a) The weight of the δ function at the origin $P_0^{(n-m)}(\frac{1}{2})$ [one should remember that in the case $L=3$ ($d\approx 2.58$) there is no δ function at the origin]. (b) The total probability of finding frustrated cells $F^{(n-m)}(\frac{1}{2})$. (c) The average number of ground states per frustrated cell $\gamma^{(n-m)}(\frac{1}{2})$; in the case $d=5$, the point $\gamma^{(n)}$ lies outside the plotted interval ($\gamma^{(n)}\approx 86$).

$$\sum_{k=n}^{n-6} \phi^{(k)} \ln \gamma^{(k)} = (0.129\,96+0.034\,52+0.011\,65+0.004\,11 + 0.001\,44+0.000\,13)N^{(n)},$$

where we have discarded the error bars due to the numerical simulations (which lead to contributions $O\sim 10^{-7}$ in the sum above). The truncation of the series generates an additional numerical uncertainty; since all terms in our series are positive, such a truncation will underestimate the total sum, leading to positive error bars. Using the fact that

$$\sum_{k=1}^{n-7} N_c^{(k)} \ln 4 = \ln 4 \sum_{k=1}^{n-7} 4^{(k-1)} = \ln 4\left(\tfrac{1}{12}4^{(n-6)} - \tfrac{1}{3}\right) \approx 4.2\times 10^{-5} N^{(n)},$$

one gets

$$\sum_{k=1}^{n} \phi^{(k)} \ln \gamma^{(k)} = (0.182\,27^{+0.000\,04}_{-0.000\,00})N^{(n)}. \qquad (3.9)$$

In the equation above, the upper error bar corresponds to the series truncation at hierarchy level $k=n-6$, whereas the lower one corresponds to the numerical simulations ($O\sim 10^{-7}$). One sees clearly that since the number of sites grows as 4^k, the contributions of the last hierarchies dominate among the others. Although nearly all cells become frustrated for $k<n-6$, their contribution to the total number of ground states is negligible, in the thermodynamic limit. Considering only the hierarchy levels $k=n$ to $k=n-6$, one gets the behavior predicted in Eq. (2.12), within a four-decimal-digit accuracy in $h(2)$; the contribution of all other hierarchies will lead to corrections in the fifth decimal digit of $h(2)$.

It should be noticed that the results above are valid for the zero-temperature *paramagnetic phase* associated with the model (2.1), defined on a DHL of fractal dimension $d=2$; in the next section, we will estimate the total number of ground states of the spin-glass phase for several DHLs with fractal dimensions greater than the lower critical dimension d_l.

IV. CASES $d>d_l$

In this section we consider DHLs with fractal dimensions $d>d_l$ (number of branches $L\geqslant 3$ in the cell of Fig. 1). It is well known that in such cases different symmetric probability distributions $P^{(n)}(J_{ij}^{(n)})$ lead to a spin-glass phase at low temperatures, which extends down to zero temperature [31,35,40,41]. Contrary to what happened in the preceding section, now we will calculate the average number of ground states associated with a spin-glass phase.

Any continuous $P^{(n)}(J_{ij}^{(n)})$ will converge, after a few iterations, to a distribution (very close to a Gaussian one [39]) with increasing variance, if $d>d_l$. Since one has zero probability of finding a branch l with $|J_{il}|=|J_{lj}|$, there will be no frustrated cells at any hierarchy level. At the zeroth hierarchy, the single bond may be either ferromagnetic (two states) or antiferromagnetic (two states) and so, $A=2$ for a given disorder configuration. Therefore, Eq. (2.6) yields only two states (related to each other by reflection symmetry),

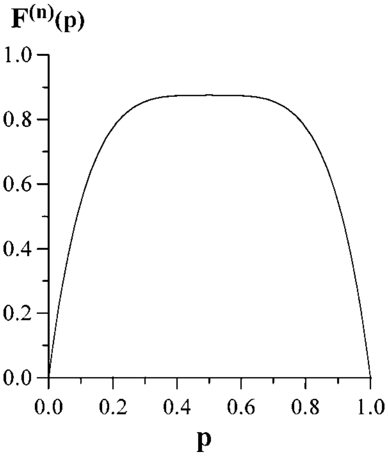

FIG. 4. The total probability of finding frustrated cells, for the DHL of fractal dimension $d=3$, at hierarchy level $k=n$, as a function of the initial ferromagnetic coupling weight p of the bimodal distribution $P^{(n)}(J_{ij}^{(n)})$.

$$[N_{\rm GS}^{(n)}]_{J^{(n)}}=2 \quad \text{for any continuous } P^{(n)}(J_{ij}^{(n)}). \tag{4.1}$$

From now on we will be restricted to the bimodal probability distribution of Eq. (2.3). The proliferation of δ functions becomes faster when the dimensionality increases; indeed, one may easily see that for an arbitrary integer dimension d, under the first renormalization transformation, one will go from the bimodal distribution of Eq. (2.3) to a composition of $2^{(d-1)}+1$ δ functions centered at $J_{ij}^{(n-1)}=-2^{(d-1)}J, -2^{(d-2)}J,\ldots, -2J, 0, 2J,\ldots, 2^{(d-2)}J, 2^{(d-1)}J$, respectively.

In the limits $p\to 1$ and $p\to 0$ the weight of the δ function corresponding to the strongest renormalized interaction will prevail among the others and so, $F_\alpha^{(k)}(p)\to 0$; since $A=2$, one gets also $[N_{\rm GS}^{(n)}]_{J^{(n)}}=2$.

Let us analyze first the interesting (and hopefully relevant to real systems) case of a $d=3$ DHL.

A. $d=3$

One gets for hierarchy level n

$$\begin{aligned} P^{(n)}(p)=&8p^7(1-p)+24p^6(1-p)^2+56p^5(1-p)^3\\ &+48p^4(1-p)^4+56p^3(1-p)^5+24p^2(1-p)^6\\ &+8p(1-p)^7, \end{aligned} \tag{4.2}$$

which is represented in the plot of Fig. 4. Similarly to what happened in the case $d=2$, one notices that the maximum probability of finding frustrated cells occurs within a wide range of values of p around $p=\frac{1}{2}$. The renormalized probability distribution (hierarchy level $k=n-1$) is composed by five δ's,

$$\begin{aligned} P^{(n-1)}(J_{ij}^{(n-1)})=&P_{4J}^{(n-1)}(p)\,\delta(J_{ij}^{(n-1)}-4J)+P_{2J}^{(n-1)}\\ &\times(p)\,\delta(J_{ij}^{(n-1)}-2J)+P_0^{(n-1)}(p)\,\delta(J_{ij}^{(n-1)})\\ &+P_{-2J}^{(n-1)}(p)\,\delta(J_{ij}^{(n-1)}+2J)+P_{-4J}^{(n-1)}\\ &\times(p)\,\delta(J_{ij}^{(n-1)}+4J), \end{aligned} \tag{4.3}$$

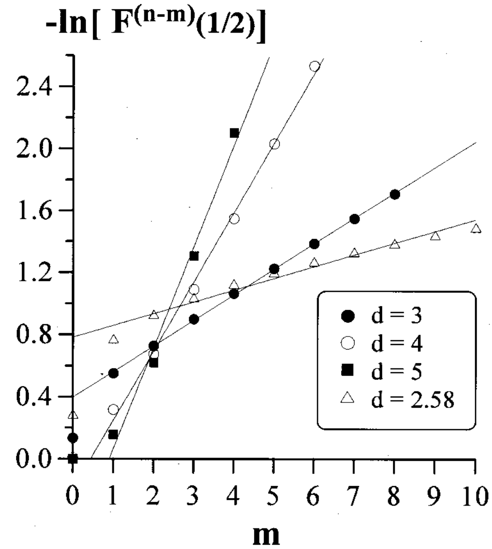

FIG. 5. A semilog plot of the data in Fig. 3(b) for DHLs of fractal dimensions $d>d_l$.

where

$$\begin{aligned} P_{4J}^{(n-1)}(p)=&p^8+4p^6(1-p)^2+6p^4(1-p)^4+4p^2(1-p)^6\\ &+(1-p)^8, \end{aligned} \tag{4.4a}$$

$$\begin{aligned} P_{2J}^{(n-1)}(p)=&8p^7(1-p)+24p^5(1-p)^3+24p^3(1-p)^5\\ &+8p(1-p)^7, \end{aligned} \tag{4.4b}$$

$$P_0^{(n-1)}(p)=24p^6(1-p)^2+48p^4(1-p)^4+24p^2(1-p)^6, \tag{4.4c}$$

$$P_{-2J}^{(n-1)}(p)=32p^5(1-p)^3+32p^3(1-p)^5, \tag{4.4d}$$

$$P_{-4J}^{(n-1)}(p)=16p^4(1-p)^4. \tag{4.4e}$$

As mentioned above, the limits $p\to 1$ and $p\to 0$ are dominated by the strongest renormalized interaction [see $P_{4J}^{(n-1)}(p)$ in Eq. (4.4a)] and one gets the trivial result $[N_{\rm GS}^{(n)}]_{J^{(n)}}=2$.

For hierarchy level $k=n-2$ one gets 17 δ functions centered at $J_{ij}^{(n-2)}=-16J,-14J,\ldots,-2J,0,2J,\ldots,14J,16J$, respectively.

Let us now restrict ourselves to $p=\frac{1}{2}$. Contrary to what happened for $d=2$, our numerical simulations indicate that the weight of the δ function centered at $J_{ij}^{(k)}=0$ decreases with each iteration [see Fig. 3(a)], characterizing a spin-glass phase. Therefore, the proliferation of δ functions occurs in such a way that, at each renormalization step, the ones corresponding to $J_{ij}^{(k)}\neq 0$ will become more and more important. As a consequence of this, the probability of finding frustrated cells decreases, as exhibited in Fig. 3(b). Indeed, such results lead to an exponential decay, as shown in Fig. 5,

$$F^{(n-m)}(1/2)\sim\exp[-C(d)m], \tag{4.5}$$

with $C(3)=0.164\pm 0.001$.

The average number of ground states $\gamma^{(n-m)}(\frac{1}{2})$ [see Fig. 3(c)] shows a little increase for $m=1$ (a direct consequence of the appearance of the δ at the origin), but decreases slowly for $m>1$. Such a slow decrease reflects the fact that $d=3$ is above, but close to, the lower critical dimension.

One may now use the results of the numerical simulations in order to compute the ground-state complexity of Eq. (2.11). Due to the simultaneous decay of the probability of finding frustrated cells $F^{(k)}(\frac{1}{2})$, the number of ground states per frustrated cell $\gamma^{(k)}(\frac{1}{2})$, and mainly due to the rapid reduction of the number of sites of the lattice $N^{(k)}$, as the hierarchy level k decreases, only the last hierarchies contribute significantly to the ground-state complexity. Indeed, one gets

$$\sum_{m=0}^{n-1} \phi^{(n-m)} \ln \gamma^{(n-m)} = (0.866\,45 \pm 0.000\,05) N_c^{(n)} + (0.619\,58 \pm 0.000\,08) N_c^{(n-1)} + 0.514\,79 N_c^{(n-2)} + 0.429\,02 N_c^{(n-3)} + 0.358\,89 N_c^{(n-4)} + \cdots .$$

In the limit $n \gg 1$ one may use Eq. (2.2c) to get

$$\sum_{m=0}^{n-1} \phi^{(n-m)} \ln \gamma^{(n-m)} = [(0.189\,54 \pm 0.000\,01) + 0.016\,94 + 0.001\,76 + 0.000\,18 + 0.000\,02 + \cdots] N^{(n)},$$

where the error bars due to simulations at hierarchy $k=n-1$ were discarded (yield contributions $O \sim 10^{-6}$ in the sum above). By truncating the series at $k=n-3$, one gets

$$\sum_{k=1}^{n} \phi^{(k)} \ln \gamma^{(k)} = (0.208\,42^{+0.000\,03}_{-0.000\,01}) N^{(n)}, \tag{4.6}$$

where the upper error bar takes into account both series truncation and numerical simulations, whereas the lower one corresponds to the numerical simulations only. The result above yields the exponential increase of Eq. (2.12) with $h(3) = 0.208\,42^{+0.000\,03}_{-0.000\,01}$. One sees that only the last four hierarchies contributed significantly to the above estimate of $h(3)$. The dominant contributions of the last hierarchies will become even more pronounced for higher dimensionalities, as we will see next.

B. $d=4$ and 5

As mentioned before, the proliferation of δ functions becomes faster as d increases and as a consequence, the probability of finding frustrated cells exhibits the exponential decay of Eq. (4.5) with higher values of the amplitude $C(d)$; Fig. 5 yields the values of $C(d)$ presented in Table I. One may now use the results of the numerical simulations to estimate the average number of ground states,

TABLE I. The quantities $C(d)$ and $h(d)$ which characterize, respectively, the exponential decay of the probability of finding frustrated cells with the renormalization step, for $d>d_l$ [Eq. (4.5)], and the exponential dependence of the number of ground states on the total number of sites of the DHL [Eq. (2.12)]. In the third column, the case $d=2$ is qualitatively distinct from those with $d>d_l$; $h(d=2)$ $[h(d>d_l)]$ refers to the ground states of a paramagnetic (spin-glass) phase. The error bars in $C(d)$ take into account uncertainties of numerical simulations, as well as of the linear fits in Fig. 5. The upper error bars in $h(d)$ take into account both series truuncation (which, in our case, always leads to positive uncertainties), as well as numerical simulations; the lower oncs refer to numerical simulations only.

d	$C(d)$	$h(d)$
2.00		$0.182\,27^{+0.000\,04}_{-0.000\,00}$
2.58	0.072 ± 0.004	$0.162\,29^{+0.000\,02}_{-0.000\,01}$
3.00	0.164 ± 0.001	$0.208\,42^{+0.000\,03}_{-0.000\,01}$
4.00	0.446 ± 0.013	$0.244\,51^{+0.000\,03}_{-0.000\,02}$
5.00	0.651 ± 0.052	$0.272\,86^{+0.000\,05}_{-0.000\,01}$

$$\sum_{m=0}^{n-1} \phi^{(n-m)} \ln \gamma^{(n-m)} = (2.014\,45 \pm 0.000\,16) N_c^{(n)} + 1.112\,30 N_c^{(n-1)} + 0.632\,33 N_c^{(n-2)} + 0.372\,70 N_c^{(n-3)} + \cdots \quad (d=4),$$

$$\sum_{m=0}^{n-1} \phi^{(n-m)} \ln \gamma^{(n-m)} = (4.456\,21 \pm 0.000\,24) N_c^{(n)} + 1.612\,68 N_c^{(n-1)} + 0.665\,11 N_c^{(n-2)} + \cdots \quad (d=5),$$

where we have written the error bars due to numerical simulations for hierarchies $k=n$ only. Using Eq. (2.2c) one gets

$$\sum_{m=0}^{n-1} \phi^{(n-m)} \ln \gamma^{(n-m)} = [(0.236\,07 \pm 0.000\,02) + 0.008\,15 + 0.000\,29 + 0.000\,01 + \cdots] N^{(n)} \quad (d=4),$$

$$\sum_{m=0}^{n-1} \phi^{(n-m)} \ln \gamma^{(n-m)} = [(0.269\,81 \pm 0.000\,01) + 0.003\,05 + 0.000\,04 + \cdots] N^{(n)} \quad (d=5).$$

Truncating the series at $k=n-2$ $(d=4)$ and $k=n-1$ $(d=5)$, one gets

$$\sum_{k=1}^{n} \phi^{(k)} \ln \gamma^{(k)} = (0.244\,51^{+0.000\,03}_{-0.000\,02}) N^{(n)} \quad (d=4), \tag{4.7}$$

$$\sum_{k=1}^{n} \phi^{(k)} \ln \gamma^{(k)} = (0.272\,86^{+0.000\,05}_{-0.000\,01}) N^{(n)} \quad (d=5), \tag{4.8}$$

leading to the values of $h(d)$ given in Table I. In both cases above, the hierarchy level n completely dominates the total number of ground states; the $(n-1)$th hierarchy leads to corrections in the third decimal digit of $h(d)$.

C. $d=2.58$

It is well known that the lower critical dimension of Ising spin glasses on DHLs is $d_l \approx 2.5$ [39,40]; curiously, this value is in good agreement with the d_l estimates for Bravais lattices [41]. Therefore, the case of a unit cell of Fig. 1 with three branches ($d \approx 2.58$) is slightly above d_l; this particular DHL allows one to study a spin-glass model practically *at* its lower critical dimension. This is the case for which the probability of finding frustrated cells exhibits the slower decrease with the renormalization step, as shown in Figs. 3(b) and 5, yielding a small value of $C(d)$ (see Table I). In addition to that, the average number of ground states per frustrated cell remains nearly unchanged under renormalization, as can be seen in Fig. 3(c). The results of the numerical simulations yield

$$\begin{aligned}\sum_{m=0}^{n-1} \phi^{(n-m)} \ln \gamma^{(n-m)} = {} & (0.519\,87 \pm 0.000\,04) N_c^{(n)} \\ & + (0.330\,52 \pm 0.000\,02) N_c^{(n-1)} \\ & + 0.284\,80 N_c^{(n-2)} + 0.256\,14 N_c^{(n-3)} \\ & + 0.234\,71 N_c^{(n-4)} + 0.217\,59 N_c^{(n-5)} \\ & + \cdots,\end{aligned}$$

and after using Eq. (2.2c),

$$\begin{aligned}\sum_{m=0}^{n-1} \phi^{(n-m)} \ln \gamma^{(n-m)} = {} & [(0.144\,41 \pm 0.000\,01) + 0.015\,30 \\ & + 0.002\,20 + 0.000\,33 + 0.000\,05 \\ & + 0.000\,01 + \cdots] N^{(n)}.\end{aligned}$$

Truncating the series at hierarchy level $k=n-4$, one gets

$$\sum_{k=1}^{n} \phi^{(k)} \ln \gamma^{(k)} = (0.162\,29^{+0.000\,02}_{-0.000\,01}) N^{(n)},$$

leading to the value of $h(d)$ in Table I.

Since all numerical simulations, which were possible to be compared with analytic calculations, yielded agreement up to three decimal digits, we expect that the values presented in Table I should represent precise estimates for DHLs. In all cases investigated, the average number of ground states shows an exponential dependence on the number of sites. Such a result is in agreement with the ground-state estimates for the infinite-range-interaction model [7,8].

V. CONCLUSION

We have calculated, by means of both analytical methods and numerical simulations, the total number of ground states for short-range Ising spin glasses on diamond hierarchical lattices of fractal dimensions $d=2$, 3, 4, 5, and 2.58. For continuous probability distributions, the number of ground states is finite in the thermodynamic limit. The maximum degeneracy occurs in the case of a bimodal ($\pm J$) distribution, for a wide range of values of the initial ferromagnetic coupling weight around $p=\frac{1}{2}$, where the average number of ground states at hierarchy level n, $[N_{\rm GS}^{(n)}]_{J^{(n)}}$, diverges with the corresponding total number of sites, $N^{(n)}$, as $[N_{\rm GS}^{(n)}]_{J^{(n)}} \sim \exp[h(d)N^{(n)}]$, in the thermodynamic limit. The result for $d=2$ corresponds to the degeneracy of the paramagnetic phase at zero temperature, since after a few iterations, the δ function centered at $J_{ij}=0$ prevails among the other ones. For fractal dimensions $d>d_l$ ($d_l \approx 2.5$ [40,41]), our results correspond to the spin-glass ground-state degeneracy, for which the amplitudes $h(d)$ have presented an increase with d. In these latter cases, under the renormalization process, the proliferation of δ functions in the coupling distribution is faster for higher values of d, in such a way that, after many iterations, one approaches a limit that resembles a continuous probability distribution; in addition to that, the total number of sites of the lattice decreases rapidly after each renormalization step. As a consequence, only a fraction of the total number of hierarchy levels contributes significantly to the ground-state degeneracy. In fact, for the cases $d=4$ and 5 the $(n-1)$th hierarchy level leads to corrections in the third decimal digit of the corresponding $h(d)$ values calculated from the highly dominant contribution of the nth hierarchy.

In all cases investigated with the symmetric bimodal probability distribution, the complexity $I=k_B \ln[N_{\rm GS}^{(n)}]_{J^{(n)}}$ turns out to be an extensive quantity. Such a property is in agreement with the mean-field picture of spin glasses. Since we have analyzed several distinct values of d, we expect the result $[N_{\rm GS}^{(n)}]_{J^{(n)}} \sim \exp[h(d)N^{(n)}]$ to hold in general diamond hierarchical lattices, with $h(d)$ increasing with the fractal dimension d ($d>d_l$). It is worth remembering that in the limit $d \to \infty$, diamond hierarchical lattices are not expected to reproduce the mean-field results; in fact, they are better approximations of Bravais lattices for low dimensionalities.

A proper understanding of the ground states in real spin glasses represents a major challenge in the physics of disordered magnets and has been the focus of attention of many workers [9–28]. Whether the present results apply, at least qualitatively in low dimensionalities, to Bravais lattices, is a question which deserves further investigation.

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